

Existence and mutiplicity of solutions to elliptic equations of fourth order on compact manifolds.

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ABSTRACT. This paper deals with a fourth order elliptic equation on compact Riemannian manifolds. We establish the existence of solutions to the equation with critical Sobolev growth which is the subject of the first theorem. In the second one, we prove the multiplicity of solutions in the subcritical case.

1. Introduction

Let (M, g) be a Riemannian compact smooth n - manifold $n \geq 5$ with the metric g , we let $H_2^2(M)$ be the standard Sobolev space which is the completion of the space

$$C_2^2(M) = \left\{ u \in C^\infty(M) : \|u\|_{2,2} < +\infty \right\}$$

with respect to the norm $\|u\|_{2,2} = \sum_{l=0}^2 \|\nabla^l u\|_2$.

Let H_2 be the space H_2^2 endowed with the equivalent norm

$$\|u\|_{H_2} = \left(\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \right)^{\frac{1}{2}}.$$

where, $\Delta(u) = -\text{div}(\nabla u)$, denotes the Riemannian laplacian.

First we investigate solutions of the critical equation

$$(1) \quad \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)|u|^{N-2}u$$

where a , h and f are smooth functions on M and $N = \frac{2n}{n-4}$ is the critical exponent. Next, we establish the existence of at least two solutions of the subcritical equation

$$(2) \quad \Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)|u|^{q-2}u$$

where $2 < q < N$.

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The function f involved in the nonlinearity is of changing sign which makes the analysis more difficult than the case where f is of constant sign.

The equation (1) has a geometric roots, in fact while the conformal Laplacian

$$L_g(u) = \Delta u + \frac{n-2}{4(n-1)}R$$

where R is the scalar curvature of the metric g is associated to the scalar curvature; the Paneitz operator as discovered by Paneitz ([10]) on 4-dimension manifolds and extended by Branson ([3]) to higher dimensions ($n \geq 5$) reads as

$$PB_g(u) = \Delta^2 u + \operatorname{div}\left(-\frac{(n-2)^2+4}{2(n-1)(n-2)}R.g + \frac{4}{n-2}\operatorname{Ric}\right)du + \frac{n-4}{2}Q^n u$$

where Ric is the Ricci curvature of g and where

$$Q^n = \frac{1}{2(n-1)}\Delta R + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R^2 - \frac{2}{(n-2)^2}|\operatorname{Ric}|^2$$

is associated to the notion of Q -curvature, good references on the subject are Chang [5] and Chang-Yang [6]. When the manifold (M, g) is Einstein, the Paneitz-Branson operator has constant coefficients. It expresses as

$$PB_g = \Delta^2 u + \alpha \Delta u + au$$

with

$$\alpha = \frac{n^2-2n-4}{2n(n-1)}R \quad \text{and} \quad a = \frac{(n-4)(n^4-4)}{16n(n-1)^2}R^2$$

and this operator is a special case of what it is usually referred as a Paneitz-Branson type operator with constant coefficients.

Since 1990 many results have been established for precise functions a , h and f . D.E. Edmunds, D. Fortunato, E. Jannelli ([8]) proved for $n \geq 8$ that if $\lambda \in (0, \lambda_1)$, with λ_1 is the first eigenvalue of Δ^2 on the euclidean open ball B , the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B \end{cases}$$

has a non trivial solution.

In 1995, R. Van der Vorst ([12]) obtained the same results as D.E. Edmunds, D. Fortunato, E. Jannelli. when applied to the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded set of R^n and moreover he showed that the solution is positive

In ([7]) D.Carafta studied the equation (1) in the case $f(x) = \text{constant}$; and in the particular case where the functions $a(x)$ and $h(x)$ are precise constants she obtained the existence of positive regular solutions.

In the case of second order equation related to the prescribed scalar curvature, that is

$$(3) \quad \Delta u + \frac{n-2}{4(n-1)} Ru = f u^{2^*-1}$$

where $2^* = \frac{2n}{n-2}$. A. Rauzy([11]) stated, in the case where the scalar curvature R of the manifold (M, g) is a negative constant and f is a changing sign function, the following results.

Let f be a C^∞ function on M , $f^- = -\inf(f, 0)$, $f^+ = \sup(f, 0)$ and

$$\lambda_f = \inf_{u \in A} \frac{\int_M |\nabla u|^2 dv_g}{\int_M u^2 dv_g}$$

where $A = \{u \in H_1^2(M), u \geq 0, u \not\equiv 0 \text{ s.t. } \int_M f^- u dv_g = 0\}$,
and $\lambda_f = +\infty$ if $A = \emptyset$.

Theorem 1' (critical case) *There is a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if $f \in C^\infty$ on M fulfills the following conditions*

$$(1') \quad |R| < \frac{4(n-1)}{n-2} \lambda_f$$

$$(2') \quad \frac{\sup f^+}{\int f^-} < C.$$

The equation (3) admits a positive solution. (R is negative constant and f is a changing sign function).

Theorem 2' (subcritical case) *For every C^∞ function f on M there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions*

$$(1'') \quad |R| < \frac{4(n-1)}{n-2} \lambda_f$$

$$(2'') \quad \frac{\sup f^+}{\int f^-} < C$$

$$(3'') \quad \sup f > 0.$$

Then the equation $\Delta_g u + Ru = f u^{q-1}$, $q \in]2, 2^[$ (R is strictly and f is a changing sign function) admits two nontrivial distinct solutions.*

More recently [2] we have extended the work of Rauzy to the case of the so called generalized prescribed scalar curvature type equation

$$(4) \quad \Delta_p u + a u^{p-1} = f u^{p^*-1}$$

where $p^* = \frac{np}{n-p}$, $\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator on a compact manifold M of dimension $n \geq 3$ with negative scalar curvature, $p \in (1, n)$, $u \in H_1^p(M)$ is a positive function, f is a changing sign function and a is a negative constant. Let

$$\lambda_f = \inf_{u \in A} \frac{\int_M |\nabla u|^p dv_g}{\int_M u^p dv_g}$$

where $A = \{u \in H_1^p(M), u > 0, u \not\equiv 0 \text{ s.t. } \int_M f^- u dv_g = 0\}$,
and $\lambda_f = +\infty$ if $A = \emptyset$

THEOREM 1. (*Critical case*) *There is a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if $f \in C^\infty$ on M fulfills the following conditions*

- (i) $|a| < \lambda_f$
- (ii) $(\sup f^+ / \int f^- dv_g) < C$
- (iii) $\sup f > 0$.

Then the equation (4) has a positive solution of class $C^{1,\alpha}(M)$.

THEOREM 2. (*Subcritical case*) *For every C^∞ -function on M there is a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if f fulfills the following conditions*

- (i) $|a| < \lambda_f$
- (ii) $(\sup f^+ / \int f^- dv_g) < C$
- (iii) $\sup f > 0$.

Then the subcritical equation

$$\Delta_p u + a u^{p-1} = f u^{q-1} \quad q \in]p, p^*[$$

has at least two non trivial positive solutions of class $C^{1,\alpha}(M)$.

For a, f , C^∞ -functions M , we let

$$\lambda_{a,f} = \inf_{u \in A} \frac{\int_M (\Delta u)^2 dv_g - \int_M a |\nabla u|^2 dv_g}{\int_M u^2 dv_g}$$

where $A = \{u \in H_2, u \not\equiv 0 \text{ s. t. } \int_M f^- |u| dv_g = 0\}$, and

$$\lambda_f = +\infty \quad \text{if} \quad A = \emptyset$$

In this paper we state the following results

THEOREM 3. *Let a, h be C^∞ negative functions on M . For every C^∞ function f on M with $\int_M f^- > 0$ there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions:*

- (i) $|h(x)| < \lambda_{a,f}$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$

the critical equation

$$\Delta^2 u + \nabla^i (a \nabla_i u) + h u = f |u|^{N-2} u$$

has a solution of class $C^{4,\alpha}$, for some $\alpha \in (0,1)$, with negative negative energy.

THEOREM 4. *Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions*

- (i) $|h(x)| < \lambda_{a,f}$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$
- (iii) $\sup f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i(a \nabla_i u) + hu = f |u|^{q-2} u, \quad q \in]2, N[$$

has at least two distinct solutions of class $C^{4,\alpha}$, for some $\alpha \in (0, 1)$.

2. Preliminaries

We suppose without lose of generality that the Riemannian manifold (M, g) is of volume equals to 1. Since it is equivalent to solve the equation (1) with f or αf (α a real number $\neq 0$), we consider the functional F_q defined on H_2 by

$$F_q(u) = \|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g + \int_M hu^2 dv_g - \int_M f |u|^q dv_g, \quad q \in]2, N[$$

and set

$$B_{k,q} = \left\{ u \in H_2(M), \quad \|u\|_q^q = k \right\}$$

where k is some constant. Let

$$\mu_{k,q} = \inf_{u \in B_{k,q}} F_q(u),$$

we state

PROPOSITION 1. *The infimum $\mu_{k,q}$ is attained .*

PROOF. We have

$$(5) \quad F_q(u) \geq \|\Delta u\|_2^2 - \|a_+\|_\infty \|\nabla u\|_2^2 + k^{\frac{2}{q}} \min_{x \in M} h(x) - k \max_{x \in M} f(x).$$

where $a_+(x) = \max[a(x), 0]$ and $\|\cdot\|_\infty$ is the supremum norm.

The following formula is well known on compact manifolds

$$\|\nabla^2 u\|_2^2 \leq \|\Delta u\|_2^2 - \int_M Ric_{ij} \nabla u_i \nabla u_j dv_g$$

$$(6) \quad \leq \|\Delta u\|_2^2 + \beta \|\nabla u\|_2^2.$$

where β is some constant. As it is shown in ([1] p.93), for any $\eta > 0$, there exists a constant $C(\eta)$ depending on η such that

$$(7) \quad \|\nabla u\|_2^2 \leq \eta \|\nabla^2 u\|_2^2 + C(\eta) \|u\|_2^2$$

Plugging (6) in (7), we get

$$(8) \quad \|\nabla u\|_2^2 \leq \eta \|\Delta u\|_2^2 + \eta\beta \|\nabla u\|_2^2 + C(\eta) \|u\|_2^2$$

and choosing η such that $\eta\beta \leq \frac{1}{2}$, we obtain

$$(9) \quad \|\nabla u\|_2^2 \leq 2\eta \|\Delta u\|_2^2 + 2C(\eta) \|u\|_2^2.$$

The inequality(5) reads then

$$F_q(u) \geq \|\Delta u\|_2^2 (1 - 2\eta \|a_+\|_\infty)$$

$$+ k^{\frac{2}{q}} \left(\min_{x \in M} h(x) - 2C(\eta) \|a_+\|_\infty \right) - k \max_{x \in M} f(x)$$

and then, with η small enough, we have

$$1 - 2\eta \|a_+\|_\infty = \alpha > 0$$

so

$$(10) \quad F_q(u) \geq \alpha \|\Delta u\|_2^2 + C_1$$

where α is some positive constant and C_1 is a constant independent of u . Let (u_j) be a minimizing sequence of the functional F_q in $B_{k,q}$; so for j sufficiently large $F_q(u_j) \leq \mu_{k,q} + 1$ and by (10), we get

$$\|\Delta u_j\|_2^2 \leq \frac{1}{\alpha} (\mu_{k,q} + 1 - C_1).$$

By formula(9) and the fact

$$\|u_j\|_2^2 \leq k^{\frac{2}{q}},$$

we obtain that $\|\nabla u_j\|_2^2$ is bounded. It follows that the sequence (u_j) is bounded in H_2 . Consequently u_j converges weakly in H_2 , the compact embedding of H_2 in L_q and the unicity of the weak limit allow us to claim that there is a subsequence of (u_j) still denoted (u_j) such that

$$u_j \rightarrow u \text{ strongly in } L^s \text{ for any } s < N$$

$$\nabla u_j \rightarrow \nabla u \text{ strongly in } L^2$$

and

$$\|u\|_{H_2} \leq \liminf_j \|u_j\|_{H_2}.$$

Consequently

$$F_q(u) = \mu_{k,q} \text{ with } \|u\|_q^q = k$$

and u fulfills

$$\begin{aligned} \int_M \Delta u \cdot \Delta v dv_g - \int_M a(x) \nabla^i u \cdot \nabla_i v dv_g + \int_M h(x) u v dv_g \\ - \frac{q}{2} \int_M f(x) |u|^{q-2} u v dv_g = \lambda_{k,q} \int_M |u|^{q-2} u v dv_g \end{aligned}$$

for any $v \in H_2$; where $\lambda_{k,q}$ is the Lagrange multiplier.

So u is a weak solution of the equation

$$(11) \quad \Delta^2 u + \nabla^i (a \nabla_i u) + hu = \left(\lambda_{k,q} + \frac{q}{2} f \right) |u|^{q-2} u.$$

Using the bootstrap method, we show that $u \in L^s(M)$ for any s , so $P(u) = \Delta^2 u + \nabla^i (a \nabla_i u) + hu \in L^s(M)$ for any s and since P is a fourth order elliptic operator, it follows by a well known regularity theorem that $P(u) \in C^{0,\alpha}(M)$ for some $\alpha \in (0, 1)$. Then $u \in C^{4,\alpha}(M)$. \square

PROPOSITION 2. $\mu_{k,q}$ is continuous as a function of the argument k .

PROOF. For any $k, l \in R^+$, let u and v be two functions of norm 1 in L^q such that $F_q(k^{\frac{1}{q}}u) = \mu_{k,q}$ and $F_q(l^{\frac{1}{q}}v) = \mu_{l,q}$.

Then

$$\begin{aligned} \mu_{l,q} - \mu_{k,q} &= F_q(l^{\frac{1}{q}}v) - F_q(k^{\frac{1}{q}}v) + F_q(k^{\frac{1}{q}}v) - \mu_{k,q} \\ &= F_q(k^{\frac{1}{q}}v) - \mu_{k,q} \\ &+ (l^{\frac{2}{q}} - k^{\frac{2}{q}}) \left(\|\Delta v\|_2^2 - \int_M a |\nabla v|^2 dv_g + \int_M h v^2 dv_g \right) \\ &- (l - k) \int_M f |v|^q dv_g. \end{aligned}$$

Consequently

$$\liminf_{l \rightarrow k} (\mu_{l,q} - \mu_{k,q}) \geq F_q(k^{\frac{1}{q}}v) - \mu_{k,q}$$

and by the definition of $\mu_{k,q}$, we get

$$(12) \quad \liminf_{l \rightarrow k} (\mu_{l,q} - \mu_{k,q}) \geq 0.$$

By writing

$$\begin{aligned} \mu_{l,q} - \mu_{k,q} &= \mu_{l,q} - F_q(l^{\frac{1}{q}}u) + F_q(l^{\frac{1}{q}}u) - F_q(k^{\frac{1}{q}}u) \\ &= \mu_{l,q} - F_q(l^{\frac{1}{q}}u) \\ &+ (l^{\frac{2}{q}} - k^{\frac{2}{q}}) \left(\|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g + \int_M h u^2 dv_g \right) \\ &- (l - k) \int_M f |u|^q dv_g \end{aligned}$$

we get

$$\limsup_{l \rightarrow k} (\mu_{l,q} - \mu_{k,q}) \leq 0$$

and taking into account of (12), we obtain

$$\lim_{l \rightarrow k} \mu_{l,q} = \mu_{k,q}.$$

□

3. Some useful lemmas

First, we quote the following lemma stated by D.Caraffa in ([7])

LEMMA 1. *Let M be a Riemannian compact manifold with dimension $n \geq 5$. For any $\epsilon > 0$ there is a constant $A(\epsilon)$ such that for any $u \in H_2$ $\|u\|_N^2 \leq K(n, 2)^2(1 + \epsilon) \|\Delta u\|_2^2 + A(\epsilon) \|u\|_2^2$ with $K(n, 2)^{-2} = \pi^2 n(n-4)(n^2 - 4)\Gamma(\frac{n}{2})^{\frac{4}{n}} \Gamma(n)^{-\frac{4}{n}}$.*

Let $\beta > 0$ the constant appearing in the inequality (6), σ any positive real number and $C(\sigma)$ a positive constant as in the inequality (7). Let $\|a_+\|_\infty = \max_{x \in M} a_+(x)$, where $a_+(x) = \max(a(x), 0)$, and take σ small enough so that $1 - \sigma(\|a_+\|_\infty + \beta) > 0$. Denote also by $\|h\|_\infty = \sup_{x \in M} |h(x)|$ the supremum norm.

As in ([11]), we define the quantities,

$$\lambda_{a,f,\eta,q} = \inf_{u \in A(\eta,q)} \frac{\|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g}{\|u\|_2^2}$$

with

$$A(\eta, q) = \left\{ u \in H_2 : \|u\|_q = 1, \int_M f^- |u|^q dv_g = \eta \int_M f^- dv_g \right\}$$

for a real $\eta > 0$,

and

$$\lambda'_{a,f,\eta,q} = \inf_{u \in A'(\eta,q)} \frac{\|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g}{\|u\|_2^2}$$

where

$$A'(\eta, q) = \left\{ u \in H_2 : \|u\|_q = 1, \int_M f^- |u|^q dv_g \leq \eta \int_M f^- dv_g \right\}.$$

The following facts which are proven in ([11]), for the Laplacian operator remain valid in the case of the bi-Laplacian operator: $\lambda'_{a,f,\eta,q}$ is a decreasing function with respect to η , bounded by $\lambda_{a,f}$ and $\lambda_{a,f,\eta,q} = \lambda'_{a,f,\eta,q}$, so $\lambda_{a,f,\eta,q}$ is also a decreasing function with respect to η , and bounded by $\lambda_{a,f}$.

Now, we will study $\lambda_{a,f,\eta,q}$, to do so, we distinguish (as it is done in ([11])) the case where the set $\{x \in M : f(x) \geq 0\}$ is positive with respect to Riemannian and the one where the set is negligible.

Case $\text{meas}(\{x \in M : f(x) \geq 0\}) > 0$

LEMMA 2. For any $q \in]2, N[$, $\lambda_{a,f,\eta,q}$ goes to $\lambda_{a,f}$ whenever η goes to zero.

PROOF. $\lambda_{a,f,\eta,q}$ is attained by a family of functions labelled $v_{\eta,q}$. The functions $v_{\eta,q}$ indexed by η are bounded in H_2^2 : since

$$\|v_{\eta,q}\|_2^2 \leq \|v_{\eta,q}\|_q^2 \text{Vol}(M)^{1-\frac{2}{q}} = 1$$

and

$$\begin{aligned} \|\Delta v_{\eta,q}\|_2^2 - \|a_+\|_\infty \|\nabla v_{\eta,q}\|_2^2 &\leq \lambda_{a,f,\eta,q} \|v_{\eta,q}\|_2^2 \\ &\leq \lambda_{a,f} \|v_{\eta,q}\|_2^2 \leq \lambda_{a,f} \end{aligned}$$

By formula(9), for a well chosen $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that

$$\|\nabla v_{\eta,q}\|_2^2 \leq 2\varepsilon \|\Delta v_{\eta,q}\|_2^2 + 2C(\varepsilon) \|v_{\eta,q}\|_2^2$$

so

$$\|\Delta v_{\eta,q}\|_2^2 \leq \lambda_f + \|a_+\|_\infty \|\nabla v_{\eta,q}\|_2^2$$

$$\leq \lambda_f + 2 \|a_+\|_\infty \left(\varepsilon \|\Delta v_{\eta,q}\|_2^2 + C(\varepsilon) \|v_{\eta,q}\|_2^2 \right)$$

and

$$\|\Delta v_{q,\eta}\|_2^2 (1 - 2\varepsilon \|a_+\|_\infty) \leq \lambda_f + 2 \|a_+\|_\infty C(\varepsilon).$$

By choosing $\varepsilon > 0$ small enough such that

$$1 - 2\varepsilon \|a_+\|_\infty > 0$$

we get that

$$\|\Delta v_{q,\eta}\|_2^2 \leq C'(\lambda_f, \|a_+\|_\infty, \varepsilon)$$

where $C'(\lambda_f, \|a_+\|_\infty, \varepsilon)$ is a constant depending of $\lambda_f, \|a_+\|_\infty, \varepsilon$.

$$\|\nabla v_{n,q}\|_2^2 \leq 2\varepsilon C(\lambda_f, \|a_+\|_\infty, \varepsilon) + 2C(\varepsilon) \leq C'(\lambda_f, \|a_+\|_\infty, \varepsilon).$$

Consequently the sequence $(v_{n,q})$ is bounded in H^2 and we have

$$\begin{aligned} v_{q\eta} &\longrightarrow v_q \text{ weakly in } H_2^2. \\ v_{q\eta} &\longrightarrow v_q \text{ strongly in } H_2^r, \quad r = 0, 1 \\ v_{q\eta} &\longrightarrow v_q \text{ strongly in } L^q \end{aligned}$$

and

$$\|\Delta v_q\|_2^2 \leq \liminf_{\eta \rightarrow 0} \|\Delta v_{q\eta}\|_2^2$$

Also

$$\|v_q\|_q = 1.$$

On the other hand

$$\int_M f^- |v_{q\eta}|^q dv_g = \eta \int_M f^- dv_g$$

so

$$\int_M f^- |v_q|^q dv_g = 0.$$

Hence

$$v_q \in A$$

and

$$\begin{aligned} \|v_q\|_2^2 \lambda_{a,f} &\leq \|\Delta v_q\|_2^2 - \int_M a |\nabla v_q|^2 dv_g \\ &\leq \liminf_{\eta \rightarrow 0} \left(\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 dv_g \right) = \liminf_{\eta \rightarrow 0} \|v_{q\eta}\|_2^2 (\lambda_{a,f,q,\eta}) \end{aligned}$$

and since by construction

$$\lambda_{a,f} \geq \lambda_{a,f,q,\eta}$$

we get that

$$\lim_{\eta \rightarrow 0} \lambda_{a,f,q,\eta} = \lambda_{a,f}.$$

□

LEMMA 3. *Let $\varepsilon > 0$, there exists η_o such that for any $\eta < \eta_o$, there is q_η such that $\lambda_{a,f,q,\eta} \geq \lambda_f - \varepsilon$ for any $q > q_\eta$.*

PROOF. We proceed by contradiction. Suppose that there is a $\varepsilon_o > 0$, such that for any η there exists an $\eta_o < \eta$ and for any q_{η_o} there is $q > q_{\eta_o}$ with $\lambda_{a,f,q,\eta} < \lambda_f - \varepsilon$. If $v_{q\eta}$ is the function in H_2^2 which achieves $\lambda_{a,f,q,\eta}$, then

$$\lambda_{a,f,q,\eta} = \frac{\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 dv_g}{\|v_{q\eta}\|_2^2}$$

with $\|v_{q\eta}\|_q^q = 1$. For a convenable η , we choose a sequence q converging to N such that

$$\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 dv_g < \lambda_f - \varepsilon_o.$$

By the same argument as in the proof of Lemma2, we get that the sequence $v_{q\eta}$ indexed by q is bounded in H_2^2 so up to a subsequence $v_{q\eta}$ converges weakly to v_η in H_2^2 and strongly in H_2^r , $r = 0, 1$. We have

$$\|\Delta v_\eta\|_2^2 \leq \lim_{q \rightarrow N} \|\Delta v_{q\eta}\|_2^2$$

and by the strong convergence in H_2^r , $r = 0, 1$ we get

$$\|\Delta v_\eta\|_2^2 - \int_M a |\nabla v_\eta|^2 dv_g < (\lambda_f - \varepsilon_o) \|v_\eta\|_2^2.$$

By the Sobolev inequality given in the Lemma1 we have for any $\varepsilon_1 > 0$ there is a constant $A(\varepsilon) > 0$ such that

$$\begin{aligned} 1 &\leq (K_2^2 + \varepsilon_1) \|\Delta v_\eta\|_2^2 + A(\varepsilon_1) \|v_\eta\|_2^2 \\ &\leq [(K_2^2 + \varepsilon_1)\lambda_{a,f} + (K_2^2 + \varepsilon_1) \|a_+\|_\infty \|\nabla v_\eta\|_2^2 + A(\varepsilon_1)] \|v_\eta\|_2^2 \\ &\leq [(K_2^2 + \varepsilon_1)(1 + \|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)] \|v_\eta\|_2^2. \end{aligned}$$

Consequently

$$\|v_\eta\|_2^2 \geq \frac{1}{(K_2^2 + \varepsilon_1)(1 + \|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)}.$$

As in [11] we can show that

$$\int_M |v_q|^N dv_g \leq 1 \quad \text{and} \quad \int_M f^- |v_q|^N dv_g \leq \eta \int_M f^- dv_g.$$

Consider the sequence of η such that for any q_η , there is a $q > q_\eta$ such that

$$\lambda_{a,f,q,\eta} \leq \lambda_f - \varepsilon.$$

Now letting the η converging to 0, if v_q is the sequence corresponding to η , v_q is bounded in H_2^2 and

$$\|v_q\|_2^2 \geq \frac{1}{(K_2^2 + \varepsilon_1)(1 + \|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)}.$$

so v_q converges weakly to $v \neq 0$ in H_2^2 and strongly to v in H_2^r , $r = 0, 1$.

On the other hand

$$\int_M f^- |v|^N dv_g \leq \lim_{q \rightarrow N} \int_M f^- |v_q|^N dv_g \leq \eta \int_M f^- dv_g$$

then $\int_M f^- |v|^N dv_g = 0$ and $v \in A$ the domain of definition of λ_f . Hence

$$\lambda_f \leq \frac{\|\Delta v\|_2^2 - \int_M a |\nabla v|^2 dv_g}{\int_M |v|^2 dv_g}.$$

A contradiction and the lemma is proved. \square

Case meas $(\{x \in M : f(x) \geq 0\}) = 0$

First, we give the lemma equivalent to the Lemma[?]

LEMMA 4. *For any fixed positive constant R , there exists η_o such that for any $\eta < \eta_o$, there is q_η fulfilling: for any $q > q_\eta$, $\lambda_{a,f,\eta,q} \geq R$.*

PROOF. We argue by contradiction. It is easy to show that $\lambda_{a,f,q,\eta}$ is achieved by a function $v_{q,\eta}$ in H_2^2 with $\|v_{q,\eta}\|_q = 1$. Suppose that there is $\lambda_{a,f,\eta,q}$ bounded when η goes to 0. Then

$$\begin{aligned} \|\Delta v_{q,\eta}\|_2^2 - \|a_+\|_\infty \|\nabla v_{q,\eta}\|_2^2 &\leq \frac{\|\Delta v_{q,\eta}\|_2^2 - \|a_+\|_\infty \|\nabla v_{q,\eta}\|_2^2}{\|v_{q,\eta}\|_2^2} \\ &\leq \lambda_{a,f,q,\eta} < +\infty. \end{aligned}$$

and proceeding as in the proof of Lemma2 we get that the sequence $v_{q,\eta}$ indexed by η is bounded in H_2^2 . Consequently the sequence $v_{q,\eta}$ converges weakly to v_q in H_2^2 and converges strongly to v_q in H_2^r , $r = 0, 1$, and strongly to v_q in L^q as η goes to 0. $\int_M f^- |v_q|^q dv_g = 0$ which implies that $v_q = 0$ almost everywhere and $\|v_q\|_q = 1$ which are in contradiction with each other. \square

Now we give an analogue to Lemma3.

LEMMA 5. *There exists an η_o such that for any $\eta < \eta_o$ there is q_η such that for any $q > q_\eta$ we have $\lambda_{a,f,q,\eta} > |a|$.*

The proof of this lemma uses the arguments as the proofs of the previous ones so we omit it.

Using the lemmas quoted above we establish

LEMMA 6. (i) *Suppose that $\sup_M f > 0$ and $\|h\|_\infty < \lambda_{a,f}$. There exists η such that $\lambda_{a,f,q,\eta} - \|h\|_\infty = \varepsilon_o > 0$. Let $b = \frac{(1-2\sigma\|a_+\|_\infty)\varepsilon_o}{[(\varepsilon_o + \|h\|_\infty + 2\|a_+\|_\infty C(\sigma))(K_2^2 + \varepsilon) + (1-2\sigma\|a_+\|_\infty)A(\varepsilon)]}$ and $\mu = \inf(b, \|h\|_\infty + 2\|a_+\|_\infty C(\sigma))$ suppose that $\frac{\sup_M f}{\int_M f^- dv_g} < \frac{\mu\eta}{8(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma))}$, where and $K(n, 2)$, $A(\varepsilon)$ are the constants appearing in the Sobolev inequality given by(5). For any $q \in]2, N[$; there exists a non empty real interval $I_q \subset \mathbb{R}^+$ such that for every $u \in H_2(M)$ with L^q -norm $k^{\frac{1}{q}}$ in I_q we have $F_q(u) \geq \frac{1}{2}\mu k^{\frac{2}{q}}$.*

PROOF. *Case* $\sup_M f > 0$.

Let $u \in H_2$ such that $\|u\|_q^q = k$.

Putting

$$G_q(u) = \|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g + \int_M hu^2 dv_g + \int_M f^- |u|^q dv_g,$$

we get

$$G_q(u) \geq \|\Delta u\|_2^2 - \|a_+\|_\infty \|\nabla u\|_2^2 - \|h\|_\infty \|u\|_2^2 + \int_M f^- |u|^q dv_g$$

and taking account of (8), we obtain that for any sufficiently real $\sigma > 0$, there is a constant $C(\sigma) > 0$ such that

$$\begin{aligned} G_q(u) &\geq (1 - 2\sigma \|a_+\|_\infty) \|\Delta u\|_2^2 \\ &\quad - (\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) \|u\|_2^2 + \int_M f^- |u|^q dv_g. \end{aligned}$$

So if

$$\int_M f^- u^q dv_g \geq \eta k \int_M f^- dv_g$$

then

$$\begin{aligned} G_q(u) &\geq (1 - 2\sigma \|a_+\|_\infty) \|\Delta u\|_2^2 \\ (13) \quad &\quad - (\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) \|u\|_2^2 + \eta k \int_M f^- dv_g \end{aligned}$$

with $\sigma > 0$ sufficiently small so that

$$1 - 2\sigma \|a_+\|_\infty > 0.$$

Now since

$$\|u\|_2^2 \leq \|u\|_q^{\frac{2}{q}} \text{Vol}(M)^{1-\frac{2}{q}} = k^{\frac{2}{q}}$$

we get

$$\begin{aligned} G_q(u) &\geq k^{\frac{2}{q}} \left[-(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) + \eta k^{1-\frac{2}{q}} \int_M f^- dv_g \right] \\ &\geq k^{\frac{2}{q}} (\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) \left(\frac{\eta k^{1-\frac{2}{q}}}{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)} \int_M f^- dv_g - 1 \right) \end{aligned}$$

and choosing k such that

$$\frac{\eta k^{1-\frac{2}{q}}}{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)} \int_M f^- dv_g - 1 \geq 1$$

that is

$$k \geq \left[2 \frac{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}}$$

we obtain

$$G_q(u) \geq k^{\frac{2}{q}} (\|h\|_\infty + 2\|a_+\|_\infty C(\sigma))$$

denote by

$$k_{1,q} = \left[2 \frac{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}}.$$

In the case $\int_M f^- u^q dv_g < \eta k \int_M f^- dv_g$, we have

$$\|\Delta u\|_2^2 - \int_M a |\nabla u|^2 dv_g \geq \lambda_{a,f,q,\eta} \|u\|_2^2$$

so

$$\begin{aligned} G_q(u) &\geq \lambda_{f,\eta,q} \|u\|_2^2 + \int_M h u^2 dv_g + \int_M f^- |u|^q dv_g \\ &\geq (\lambda_{a,f,\eta,q} - \|h\|_\infty) \|u\|_2^2 + \int_M f^- |u|^q dv_g \end{aligned}$$

by Lemma[?] and [?] there exists η such that

$$\lambda_{a,f,\eta,q} - \|h\|_\infty = \varepsilon_o > 0.$$

Now, putting $\delta_1 + \delta_2 = \varepsilon_o$, where δ_1 and δ_2 are positive real numbers, and solving $\|u\|_2^2$ in ??

$$\|u\|_2^2 \geq \frac{1}{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)} \left[(1 - 2\sigma \|a_+\|_\infty) \|\Delta u\|_2^2 - G_q(u) + \int_M f^- |u|^q dv_g \right].$$

Consequently

$$\left(1 + \frac{\delta_2}{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)} \right) G_q(u) \geq \delta_1 \|u\|_2^2 + \frac{\delta_2}{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)} (1 - 2\sigma \|a_+\|_\infty) \|\Delta u\|_2^2$$

so

$$G_q(u) \geq \frac{\delta_1 (\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma))}{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma) + \delta_2} \|u\|_2^2 + \frac{\delta_2 (1 - 2\sigma \|a_+\|_\infty)}{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma) + \delta_2} \|\Delta u\|_2^2$$

and where σ is sufficiently small and such that $1 - 2\sigma \|a_+\|_\infty > 0$.

Or

$$G_q(u) \geq \frac{\delta_2 (1 - 2\sigma \|a_+\|_\infty)}{(\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma) + \delta_2) (K_2^2 + \varepsilon)} \left[(K_2^2 + \varepsilon) \|\Delta u\|_2^2 + \frac{\delta_1 (\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon)}{\delta_2 (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} \right]$$

where for any fixed $\varepsilon > 0$, K_2^2 is the best Sobolev constant in the embedding of $H_2^2(R^n)$ in $L^q(R^n)$.

Taking δ_1 and δ_2 such that

$$\frac{\delta_1 (\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon)}{\delta_2 (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} = 1$$

we get

$$\delta_1 = \frac{(1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)}{(\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} \varepsilon_o$$

and

$$\delta_2 = \frac{(\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon)}{(\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} \varepsilon_o.$$

Consequently

$$G_q(u) \geq \frac{\delta_2 (1 - 2\sigma \|a_+\|_\infty)}{(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma) + \delta_2) (K_2^2 + \varepsilon)} \|u\|_q^2$$

and since

$$\begin{aligned} \|h\|_\infty + 2\|a_+\|_\infty C(\sigma) + \delta_2 &= (\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) \left[1 + \frac{K_2^2 + \varepsilon}{(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} \right] \\ &= \frac{(\varepsilon_o + \|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)}{(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)} \end{aligned}$$

we get that

$$G_q(u) \geq \frac{(1 - 2\sigma \|a_+\|_\infty) \varepsilon_o}{[(\varepsilon_o + \|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)]} k^{\frac{2}{q}}$$

Letting

$$b = \frac{(1 - 2\sigma \|a_+\|_\infty) \varepsilon_o}{[(\varepsilon_o + \|h\|_\infty + 2\|a_+\|_\infty C(\sigma)) (K_2^2 + \varepsilon) + (1 - 2\sigma \|a_+\|_\infty) A(\varepsilon)]}$$

we get

$$\begin{aligned} F_q(u) &= G_q(u) - \int_M f^+ |u|^q dv_g \\ &\geq bk^{\frac{2}{q}} - \int_M f^+ |u|^q dv_g \geq bk^{\frac{2}{q}} - k \sup f = k^{\frac{2}{q}} (b - k^{1-\frac{2}{q}} \sup f). \end{aligned}$$

So if $\sup f > 0$, let $\mu = \inf (b, \|h\|_\infty + 2\|a_+\|_\infty C(\sigma))$. For any $k \geq k_{1,q}$, we have

$$F_q(u) \geq k^{\frac{2}{q}} (\mu - k^{1-\frac{2}{q}} \sup f)$$

and

$$F_q(u) \geq \frac{1}{2} \mu k^{\frac{2}{q}}$$

provided that

$$k \leq \left[\frac{\mu}{2 \sup f} \right]^{\frac{q}{q-2}}.$$

Now if we put $C_q = \frac{\eta}{8(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma))} \mu$ and suppose that $\sup f \leq C_q \int_M f^-$, we obtain that the inequality is fulfilled provided that

$$k \leq \left[\frac{4(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma))}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}} = 2^{\frac{q}{q-2}} k_{1,q}.$$

We put $k_{2,q} = 2^{\frac{q}{q-2}} k_{1,q}$.

Case $\sup f = 0$.

In this case, for any $k \geq k_{1,q}$, $F_q(u) \geq \frac{1}{2} \mu k^{\frac{2}{q}}$.

□

4. Solutions in the critical case

Now, we are going to investigate solutions of the critical equation. First we have

LEMMA 7. *For each $t > 0$, small enough, $\inf_{\|u\|_{H_2} \leq t} F_q(u) < 0$, $q \in]2, N]$.*

In fact $F_q(t) \leq t^2 (h - t^{q-2}) \int_M f dv_g$, where $h = \max_M h(x)$, and since $h < 0$, there is $t_o > 0$ small enough such that $\inf_{\|u\|_{H_2} \leq t} F_q(u) < 0$ for each $t \in]0, t_o[$.

LEMMA 8. *Let $u \in H_2$. If the L_q -norm $\|u\|_q^q = k$ goes to infinite, then $\mu_{t,q} = \inf_{\|u\|_q^q = k} F_q(u) \rightarrow -\infty$.*

PROOF. In fact since $\sup_{x \in M} f(x) > 0$ let u be a function of class C^2 with support contained in the open subset $\{x \in M : f(x) > 0\}$ of the manifold M such that $\|u\|_q^q = 1$, then $\int_M f |u|^q dv_g > 0$ and

$$F_q(ku) = k^{\frac{2}{q}} \left(\int_M ((\Delta u)^2 - a |\nabla u|^2 + hu^2) dv_g - k^{\frac{q-2}{q}} \int_M f |u|^q dv_g \right).$$

So $\lim_{k \rightarrow +\infty} F_q(ku) = -\infty$. \square

PROPOSITION 3. *Let a, h be C^∞ functions on M , with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{f^+}$ such that if f satisfies the following conditions*

- (i) $|h(x)| < \lambda_f$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$

the subcritical equation

$$\Delta^2 u_q + \nabla^i (a \nabla_i u_q) + hu_q = f |u_q|^{q-2} u_q \quad \text{with } q \in]2, N[$$

admits a $C^{4,\alpha}$, for some $\alpha \in (0, 1)$, solution u_q with negative energy.

PROOF. For any $q \in]2, N[$ and $k > 0$, let $\mu_{k,q} = \inf_{\|u\|_q^q = k} F_q(u)$. First we remark that if k is close to 0, $k > 0$, $\mu_{k,q} < 0$: indeed

$$\mu_{k,q} \leq F_q(k^{\frac{1}{q}}) = k^{\frac{2}{q}} \left(\int_M h dv_g - k^{1-\frac{2}{q}} \int_M f dv_g \right) < 0.$$

By proposition(2) the curve $k \rightarrow \mu_{k,q}$ is continuous and $\mu_{k,q}$ goes to 0, when $k \rightarrow 0$. So by Lemmas (6),(7) and (8) the curve $k \rightarrow \mu_{k,q}$ starts at 0, takes a negative minimum, say at k_q , then takes a positive maximum and goes to minus infinite. Let $l_q = k_{1,q} = \left[2 \frac{\|h\|_\infty + 2\|a\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}}$ the lower bound of the interval I_q given in the proof of Lemma(6), then

$$\mu_{k_q,q} = \inf_{\|u\|_q^q \leq l_q} F_q(u).$$

By proposition(1) the infimum $\mu_{kq,q}$ is attained by a function $v_q \in H_2$ with $\|v_q\|_q^q = k_q$, so

$$F_q(v_q) = \inf_{\|u\|_q^q \leq l_q} F_q(u).$$

Now since for any $k \in I_q$, and any $u \in H_2$ with $\|u\|_q^q = k$, $F_q(u) \geq 0$, it follows that $k_q < l_q$. So v_q is a critical point of F_q , that is for $\varphi \in H_2$

$$\begin{aligned} & \int_M \Delta v_q \Delta \varphi dv_g - \int_M a \nabla v_q \nabla \varphi dv_g + \\ & \int_M h v_q \varphi dv_g - \frac{q}{2} \int_M f |v_q|^{q-2} v_q \varphi dv_g = 0 \end{aligned}$$

then $u_q = (\frac{q}{2})^{\frac{1}{q-2}} v_q$ is a weak solution of the subcritical equation with negative energy such that

$$\|u_q\|_q^q \leq (\frac{q}{2})^{\frac{q}{q-2}} l_q.$$

Moreover, arguing as in the proof of the proposition(1), $u_q \in C^{4,\alpha}(M)$ with $\alpha \in (0, 1)$. \square

Finally, we seek for a solutions of the critical equation. Mainly we state

THEOREM 5. *Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{f^+}$ such that if f satisfies the following conditions*

- (i) $|h(x)| < \lambda_f$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$

the critical equation

$$(4.1) \quad \Delta^2 u + \nabla^i(a \nabla_i u) + hu = f |u|^{N-2} u$$

admits a $C^{4,\alpha}$, for some $\alpha \in (0, 1)$, solution u with negative energy.

PROOF. Let $(u_q)_q$ be the sequence of subcritical solutions of the equation [?] .We have already shown in the proof of Proposition[?] that

$$\|u_q\|_q^q = k_q \leq l_q = \left[2 \frac{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}}$$

and since l_q goes to $l_N = \left[2 \frac{\|h\|_\infty + 2 \|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{4}{n}}$ as q goes to N , $(u_q)_q$ is bounded in L^q , so it is in L^2 and since u_q are of negative energy then

$$\begin{aligned} \|\Delta u_q\|_2^2 & \leq \int_M a |\nabla u|^2 dv_g - \int_M h u_q^2 + \int_M f |u_q|^q \\ & \leq \|a_+\|_\infty \|\nabla u_q\|_2^2 + \|h\|_\infty \|u_q\|_q^2 + \|f\|_\infty \|u_q\|_q^q. \end{aligned}$$

Now since for any $\sigma > 0$, there exists a constant $C(\sigma)$ such that

$$\|\nabla u_q\|_2^2 \leq 2\sigma \|\Delta u_q\|_2^2 + 2C(\sigma) \|u_q\|_2^2$$

we get

$$\begin{aligned} (1 - 2\sigma \|a_+\|_\infty) \|\Delta u_q\|_2^2 &\leq (2\|a_+\|_\infty C(\sigma) + \|h\|_\infty) \|u_q\|_q^2 + \|f\|_\infty \|u_q\|_q^q \\ &\leq (2\|a_+\|_\infty C(\sigma) + \|h\|_\infty) l_q^{\frac{2}{q}} + \|f\|_\infty l_q. \end{aligned}$$

So $(u_q)_q$ is a bounded sequence in H_2 . Consequently $u_q \rightarrow v$ weakly in H_2 , up to a subsequence, we have

$$u_q \rightarrow v \text{ strongly in } L^s(M) \quad \text{for } s < N$$

$$\nabla u_q \rightarrow \nabla v \text{ strongly in } L^2$$

$$u_q(x) \rightarrow v(x) \quad \text{for a.e. } x \in M.$$

On the other hand for any $q \in]2, N[$, u_q satisfies, for any $\varphi \in H_2$

$$\begin{aligned} \int_M \Delta u_q \Delta \varphi dv_g - \int_M a \nabla^i u_q \nabla_i \varphi dv_g + \int_M h u_q \varphi dv_g \\ = \frac{q}{2} \int_M f |u_q|^{q-2} u_q \varphi dv_g \end{aligned}$$

and since the convergence of $(u_q)_q$ is weak in H_2 , it follows that for any $\varphi \in H_2$

$$\begin{aligned} \int_M \Delta u_q \Delta \varphi dv_g - \int_M a \nabla^i u_q \nabla_i \varphi dv_g + \int_M h u_q \varphi dv_g \\ (16) \quad \rightarrow \int_M \Delta v \Delta \varphi dv_g - \int_M a \nabla^i v \nabla_i \varphi dv_g + \int_M h v \varphi dv_g. \end{aligned}$$

Moreover since $u_q(x) \rightarrow v(x)$ for a.e. $x \in M$ and (u_q) is bounded in H_2 we have

$$u_q(x) |u_q(x)|^{q-2} \rightarrow v(x) |v(x)|^{N-2} \quad \text{for a.e. } x \in M$$

and

$$\left\| u_q |u_q|^{q-2} \right\|_{\frac{N}{N-1}} = \|u_q\|_{(q-1)\frac{N}{N-1}}^{q-1} \leq C_1 \|u_q\|_N^{q-1} \leq C \|u_q\|_{H_2}.$$

consequently (u_q) is bounded in $L_{\frac{N}{N-1}}$ and by a well known theorem([1]) u_q converges weakly to v in $L_{\frac{N}{N-1}}$. Now for any $\varphi \in H_2 \subset L_N$, and any smooth function f , $f \varphi \in L_N$ (the dual space of $L_{\frac{N}{N-1}}$), then

$$(17) \quad \int_M f |u_q|^{q-2} u_q \varphi dv_g \rightarrow \int_M f |v|^{N-2} v \varphi dv_g.$$

So by (16) and (17) $u = \left(\frac{N}{2}\right)^{\frac{1}{N-2}} v$ is a weak solution of the critical equation which is a negative minimum of the energy functional F_N that is $\mu_N = F_N(u)$. It remains to check that $u \neq 0$. Suppose that $u = 0$, then

for any $\epsilon > 0$, $|\mu_q| \leq \epsilon$ for any q close to N . Let k with $0 < k < \min \left[1, \frac{2b}{\eta \int f^-}, \frac{\max_{x \in M} h(x)}{\min_{x \in M} f(x)} \right]^{\frac{n}{4}}$, then

$$\begin{aligned} |\mu_q| &\geq k \int_M f dv_g - k^{\frac{2}{q}} \int_M h dv_g \\ &\geq k^{\frac{2}{q}} \left(k^{1-\frac{2}{q}} \min_{x \in M} f(x) - \max_{x \in M} h(x) \right) > 0. \end{aligned}$$

Consequently

$$\begin{aligned} \epsilon &\geq k^{\frac{2}{q}} \left(k^{1-\frac{2}{q}} \min_{x \in M} f(x) - \max_{x \in M} h(x) \right) \\ &> -k^{\frac{2}{q}} \min_{x \in M} f(x) (k^{1-\frac{2}{q}} + k^{\frac{4}{n}}) > -k \min_{x \in M} f(x) = \alpha > 0 \end{aligned}$$

a contradiction. By the bootstrap method (see [12]), we get that u is of class $C^{4,\alpha}$ for some $\alpha \in (0, 1)$. \square

5. Multiplicity of solutions in the subcritical case

First, we show that F_q , $q \in]2, N[$ satisfies the Palais-Smale condition.

LEMMA 9. *Let c be a real number, then each Palais-Smale sequence at level c for the functional F_q satisfies the Palais-Smale condition.*

PROOF. First, we show that each Palais-Smale sequence is bounded: we argue by contradiction. Suppose that there exists a sequence (u_j) such that $F_q(u_j)$ tends to a finite limit c , $F'_q(u_j)$ goes to zero and u_j to infinite in the H_2 -norm. More explicitly we have

$$\int_M \left((\Delta u_j)^2 - a |\nabla u_j|^2 + h u_j^2 \right) dv_g - \int_M f |u_j|^q dv_g \rightarrow c$$

and for each $v \in H_2$

$$\int_M \left(\langle \nabla \Delta u_j, \nabla v \rangle + a \langle \nabla u_j, \nabla v \rangle + h u_j^2 \right) dv_g - \frac{q}{2} \int_M f |u_j|^{q-1} v dv_g \rightarrow 0$$

so for any $\varepsilon > 0$ there exists a positive integer N such that for every $j \geq N$ we have

$$\left| \int_M \left((\Delta u_j)^2 - a |\nabla u_j|^2 + h u_j^2 \right) dv_g - \int_M f |u_j|^q dv_g - c \right| \leq \varepsilon$$

and

$$\left| \int_M \left(\langle \nabla \Delta u_j, \nabla v \rangle - a \langle \nabla u_j, \nabla v \rangle + h u_j^2 \right) dv_g - \frac{q}{2} \int_M f |u_j|^{q-1} v dv_g \right| \leq \varepsilon.$$

In the particular case where $v = u_j$, we get

$$\left| \int_M \left((\Delta u_j)^2 - a |\nabla u_j|^2 + h u_j^2 \right) dv_g - \frac{q}{2} \int_M f |u_j|^q dv_g \right| \leq \varepsilon.$$

Then, we obtain

$$(18) \quad \left| (q-2) \int_M (\Delta u_j)^2 - a |\nabla u_j|^2 + h u_j^2 dv_g - qc \right| \leq (q+2)\epsilon$$

and

$$(19) \quad \left| (q-2) \int_M f u_j^q - 2c \right| \leq 4\epsilon.$$

By Lemma 6, we can choose k to be an L^q -norm such that

$$\inf_{\|u\|_q^q = k} F_q(u) > 0.$$

Letting $v_j = k^{\frac{1}{q}} \frac{u_j}{\|u_j\|_q}$, we obtain from (18) and (19) that

$$(20) \quad \left| (q-2) \int_M f v_j^q dv_g - \frac{2ck^{\frac{2}{q}}}{\|u_j\|_q^2} \right| \leq 4\epsilon \frac{k^{\frac{2}{q}}}{\|u_j\|_q^2}.$$

and

$$(21) \quad \begin{aligned} & \left| (q-2) \int_M (\Delta v_j)^2 - a |\nabla v_j|^2 + h v_j^2 dv_g - qc \frac{k^{\frac{2}{q}}}{\|u_j\|_q^2} \right| \\ & \leq (q+2)\epsilon \frac{k^{\frac{2}{q}}}{\|u_j\|_q^2} \end{aligned}$$

Now since $\|v_j\|_q$ is a bounded sequence, it follows by (21) that (v_j) is bounded in H_2 . If $\|u_j\|_q$ goes to infinity, it follows from (20) and (21) that $F_q(v_j)$ goes to zero. And since $\|v_j\|_q^q = k$, we have

$$\inf_{\|u\|_q^q = k} F_q(u) \leq F_q(v_j)$$

so

$$\inf_{\|u\|_q^q = k} F_q(u) \leq 0.$$

Hence a contradiction. Then the sequence (u_j) is bounded in H_2 . Since $q < N$, the Sobolev injections are compact. Consequently the Palais-Smale condition is satisfied. \square

PROPOSITION 4. *Let a, h be C^∞ functions on M with f negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions*

- (i) $|h(x)| < \lambda_f$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$
- (iii) $\sup f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i(a \nabla_i u) + hu = f |u|^{q-2} u, \quad q \in]2, N[$$

admits a nontrivial solution of class $C^{4,\alpha}$, for some $\alpha \in (0, 1)$, with positive energy.

PROOF. Mimicking which is done in ([11]), let l_o be an L^q -norm such that $\mu_{l,q}$ is a maximum and l_1, l_2 two L^q -norms such that $\mu_{l_1,q} = \mu_{l_2,q} = 0$ with $l_1 < l_o$ and $l_2 > l_o$.

Set

$$\Gamma = \{\gamma \in C([0, 1], H_2) : \gamma(0) = u_{l_1,q}, \gamma(1) = u_{l_2,q}\},$$

and

$$\nu_q = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F_q(\gamma(t)) .$$

Arguing as in ([11]), we show that ν_q is a critical level of the functional F_q and $\nu_q \geq \mu_{l,q} > 0$. Consequently the subcritical equation (2) admits a weak solution of positive energy. This solution is in fact of class $C^{4,\alpha}$ with $\alpha \in (0, 1)$. \square

Now, by propositions (3) and (4) we obtain

THEOREM 6. Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions

- (i) $|h(x)| < \lambda_f$ for any $x \in M$
- (ii) $\frac{\sup f^+}{\int f^-} < C$
- (iii) $\sup f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i(a \nabla_i u) + hu = f |u|^{q-2} u, \quad q \in]2, N[$$

has two distinct solutions of class $C^{4,\alpha}$, for some $\alpha \in (0, 1)$.

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